Solution to Assignment 4

1. A trigonometric polynomial is $p(\cos x, \sin x)$ where $p(x,y)$ is a polynomial in two variables. Its degree is the degree of p. For instance, let $p(x,y) = x^2y - 6xy + 3y - 5$ which is of degree 3, the corresponding trigonometric polynomial is $\cos^2 x \sin x - 6\cos x \sin x + 3\sin x - 5$. Show that every finite trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

can be expressed as a trigonometric polynomial of degree n and the converse is true.

Solution. Use Euler's formula $e^{kix} = \cos kx + i \sin kx$, we see

$$(\cos x + i \sin x)^k = \cos kx + i \sin kx.$$

By binomial expansion we have

$$\sum_{j=0}^{k} C(k,j) i^j \cos^{k-j} x \sin^j x = \cos kx + i \sin kx \ ,$$

where $C(k,j)$ are the binomial coefficients. By equalling the real and imaginary parts we see $\cos kx$ and $\sin kx$ are trigonometric polynomials of degree k.

Conversely, using the substitution

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

in the trigonometric polynomial, we see that $p(\cos x, \sin x)$ can be written as a linear combination of $e^{ikx}, -n \le k \le n$. Thus it is a finite trigonometric series after we replace e^{ikx} by $\cos kx + i \sin kx$.

In conclusion, finite trigo series and trigo polynomials are the same thing.

2. Show that for two continuous, 2π-periodic functions f and g, they are identical if their Fourier series are the same. Hint: Show that $\int_{-\pi}^{\pi} (f-g)(x)p(x)dx = 0$ for all finite trigonometric series.

Solution. This problem is the same as to prove, a continuous function vanishing everywhere if its Fourier series is identically zero. We follow the hint. Since every finite trigo series is a linear combination of $\cos nx$ and $\sin nx$, the assumption implies

$$\int (f-g)p(x)dx = 0$$

for all finite trigo series. By Theorem 4.2, we can find a sequence of such functions $\{p_n\}$ such that $|f(x) - g(x) - p_n(x)| < 1/n$ for all x. It follows that

$$\begin{aligned}\int_{-\pi}^{\pi} (f(x) - g(x))^2 dx &= \int_{-\pi}^{\pi} (f(x) - g(x))(f(x) - g(x) - p_n(x))dx + \int_{-\pi}^{\pi} (f(x) - g(x))p_n(x)dx \\ &= \int_{-\pi}^{\pi} (f(x) - g(x))(f(x) - g(x) - p_n(x))dx \ .\end{aligned}$$

Therefore,

$$\int_{-\pi}^{\pi} (f(x) - g(x))^2 dx \le \int_{-\pi}^{\pi} |f(x) - g(x)||f(x) - g(x) - p_n(x)|dx \le M \times \frac{1}{n} \times 2\pi \to 0 \ ,$$

as $n \to \infty$. Here M is a bound on $\sup_x |f(x) - g(x)|$. We conclude that

$$\int_{-\pi}^{\pi} (f-g)^2 dx = 0$$

which forces $f - g \equiv 0$ by continuity.

Note. For completeness, let us show that if $\int_a^b F(x)dx = 0$ where F is a non-negative continuous function, then $F \equiv 0$. For, if not, $F(x_0) > 0$ at some x_0. By continuity we may assume x_0 belongs to the interior of the interval. We can find a small $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset [a, b]$ such that $F(x) \geq F(x_0)/2 > 0$ on this subinterval. But then

$$\begin{aligned}
\int_a^b F(x)dx &= \int_a^{x_0-\delta} F(x)dx + \int_{x_0-\delta}^{x_0+\delta} F(x)dx + \int_{x_0+\delta}^{b} F(x)dx \\
&\geq \int_{x_0-\delta}^{x_0+\delta} F(x)dx \\
&\leq \frac{F(x_0)}{2} \times 2\delta \\
&= \delta F(x_0) > 0 \ ,
\end{aligned}$$

contradiction holds. We apply this result to the previous paragraph by taking $F = (f-g)^2$.

3. Find the first twenty data for the following sequences and count how many are in the intervals $I_1 = [0, 0.25), I_2 = [0.25, 0.75)$ and $I_3 = [0.5, 1)$ respectively in each case.

$$(a) \ \ \langle n\sqrt{3} \rangle \ , \quad (b) \ \ \langle p_n\sqrt{2} \rangle \ , \quad (c) \ \ \left\langle \frac{(1+\sqrt{5})^n}{2} \right\rangle \ .$$

Here p_n is the n-th prime number ($p_1 = 2, \ p_2 = 3$, etc). What conclusion on their distribution can you draw? Try more data if you don't see the trend.

4. The Fibonacci numbers are given by the sequence $\{U_n\}$ satisfying $U_{n+1} = U_n + U_{n-1}, U_0 = 2, U_1 = 1$. Show that

$$U_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \ , \quad n \geq 0 \ .$$

Solution. By a standard induction. $n = 2$ clearly holds. Assume it holds for all $k \leq n$. We have

$$\begin{aligned}
U_{n+1} = U_n + U_{n-1} &= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \left(1 + \frac{1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \left(1 + \frac{1-\sqrt{5}}{2}\right) \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \ ,
\end{aligned}$$

done.

5. Prove that the sequence $\{\gamma_n\}$, where γ_n is the fractional part of $((1+\sqrt{5})/2)^n, n \geq 1$, is not equidistributed in $[0,1)$.

   Solution. From the previous problem we know that $((1+\sqrt{5})/2)^n \equiv -((1-\sqrt{5})/2)^n (\text{mod } 1)$. But $((1-\sqrt{5})/2)^n = (-2/(1+\sqrt{5}))^n$ forms a sequence which is positive and negative alternating and converging to 0, so the sequence it generates accumulates near 0 and 1 eventually.

6. (Optional) Show that for $\sigma \in (0,1)$, the sequence $\{< n^\sigma >\}$ is equidistributed in $[0,1)$. Hint: Prove that
$$\sum_{n=1}^{N} e^{2\pi i k n^\sigma} = O(N^\sigma) + O(N^{1-\sigma})$$
   by noting
$$\sum_{n=1}^{N} e^{2\pi k i n^\sigma} - \int_1^N e^{2\pi i k x^\sigma} dx = O\left(\sum_{n=1}^{N} n^{\sigma-1}\right) .$$

   Solution. As $e^{2\pi k n^\sigma} = \cos(2\pi k n^\sigma) + i\sin(2\pi k n^\sigma)$, we consider the real and imaginary parts separately. On each $[n, n+1]$, by the mean-value property of the integral we have $\int_n^{n+1} \cos(2\pi n x^\sigma) dx = \cos(2\pi n y^\sigma)$ for some $y \in [n, n+1]$. Therefore, by applying the mean-value theorem
$$\cos(2\pi k n^\sigma) - \int_n^{n+1} \cos(2\pi k x^\sigma) dx = \cos(2\pi k n^\sigma) - \cos(2\pi k y^\sigma) = -2\pi k \sin(2\pi k c^\sigma) \sigma c^{\sigma-1}(n-y)$$
   for some mean value c lying between y and n. We have
$$\left|\cos(2\pi k n^\sigma) - \int_n^{n+1} \cos(2\pi k x^\sigma) dx\right| \leq \left|(-2\pi k \sin(2\pi k c^\sigma))\sigma c^{\sigma-1}(n-y)\right| \leq C n^{\sigma-1} .$$
   Summing up, we have
$$\left|\sum_{n=1}^{N} \cos(2\pi k n^\sigma) - \int_1^N \cos(2\pi k x^\sigma) dx\right| \leq C \sum_{n=1}^{N} n^{\sigma-1} .$$
   Similarly we can treat the imaginary part.

   Now, by the integral test,
$$\sum_{n=1}^{N} n^{\sigma-1} \geq \int_1^{N+1} x^{\sigma-1} dx = \sigma^{-1}((N+1)^\sigma - 1) = O(N^\sigma),$$
   and
$$\sum_{n=2}^{N} n^{\sigma-1} \leq \int_1^{N+1} x^{\sigma-1} dx = O(N^\sigma) .$$
   It follows that
$$\sum_{n=1}^{N} n^{\sigma-1} = O(N^\sigma) .$$
   On the other hand,
$$\int_1^N \cos(2\pi k x^\sigma) dx = \sigma^{-1} \int_1^{N^\sigma} y^{1/\sigma - 1} \cos(2\pi k y) dy.$$

We write

$$\int_1^{N^\sigma} y^{1/\sigma-1}\cos(2\pi ky)dy = \sum_{j=k}^{M}\int_{j/k}^{(j+1)/k} y^{1/\sigma-1}\cos(2\pi ky)dy = \sum_{j=k}^{M}\int_0^{1/k}\left(z+\frac{j}{k}\right)^{1/\sigma-1}\cos(2\pi kz)dz\ ,$$

where M is the number so that $(M+1)/k$ is closest to N^σ. For each j,

$$\begin{aligned}\int_0^{1/k}\left(z+\frac{j}{k}\right)^{1/\sigma-1}\cos(2\pi kz)dz &= \left(\int_0^{1/2k}+\int_{1/2k}^{1/k}\right)\left(z+\frac{j}{k}\right)^{1/\sigma-1}\cos(2\pi kz)dz \\ &= \int_0^{1/2k}\left[\left(z+\frac{j}{k}\right)^{1/\sigma-1}-\left(z+\frac{j}{k}+\frac{1}{2k}\right)^{1/\sigma-1}\right]\cos(2\pi kz)dz\ .\end{aligned}$$

By the mean-value theorem,

$$\left(z+\frac{j}{k}\right)^{1/\sigma-1}-\left(z+\frac{j}{k}+\frac{1}{2k}\right)^{1/\sigma-1}=\left(\frac{1}{\sigma}-1\right)\left(z+\frac{j}{k}+c\right)^{\frac{1}{\sigma}-2}\frac{1}{2k}\ ,\ c\in\left(0,\frac{1}{2k}\right)\ .$$

Using this we see that

$$\int_0^{1/k}\left(z+\frac{j}{k}\right)^{1/\sigma-1}\cos(2\pi kz)dz$$

is like

$$C\int_0^{j/k}\left(z+\frac{j}{k}\right)^{\frac{1}{\sigma}-2}dz\ .$$

Therefore,

$$\begin{aligned}\left|\int_1^{N^\sigma} y^{1/\sigma-1}\cos(2\pi ky)dy\right| &= \left|\sum_{j=k}^{M}\int_0^{1/k}\left(z+\frac{j}{k}\right)^{1/\sigma-1}\cos(2\pi kz)dz\right| \\ &\le \left|C\sum_{j=k}^{M}\int_0^{1/k}\left(z+\frac{j}{k}\right)^{\frac{1}{\sigma}-2}dz\right| \\ &\le C\int_1^{N^\sigma} y^{\frac{1}{\sigma}-2}dy = CN^{1-\sigma}\ .\end{aligned}$$

We conclude that

$$\sum_{n=1}^{N} e^{2\pi ikn^\sigma} = O(N^\sigma)+O(N^{1-\sigma})$$

holds. Finally, the result comes from Weyl's criterion as $\sigma\in(0,1)$.

I did not realize that the solution is too long. You may wish to skip it.